LUSTERNIK-SCHNIRELMANN CATEGORY OF SIMPLICIAL COMPLEXES AND FINITE SPACES

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Abstract. In this paper we establish a natural definition of Lusternik-Schnirelmann category for simplicial complexes via the well known notion of contiguity. This category has the property of being homotopy invariant under strong equivalences, and it only depends on the simplicial structure rather than its geometric realization.

In a similar way to the classical case, we also develop a notion of geometric category for simplicial complexes. We prove that the maximum value over the homotopy class of a given complex is attained in the core of the complex.

Finally, by means of well known relations between simplicial complexes and posets, specific new results for the topological notion of LS-category are obtained in the setting of finite topological spaces.

Contents

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1. Introduction

Lusternik-Schnirelmann category was originally introduced as a tool for variational problems on manifolds. Nowadays it has been reformulated as a numerical invariant of topological spaces and has become an important notion in homotopy theory and many other areas [5], as well as in applications like topological robotics [8]. Many papers have appeared on this topic and the original definition has been generalized in a number of different ways. For simplicial complexes and simplicial maps, the notion of *contiguity* is considered as the discrete version of homotopy. However, although these notions are classical ones, the corresponding theory of LS-category is missing in the literature. This paper can be considered as a first step in this direction.

Still more important, finite simplicial complexes play a fundamental role in the so-called theory of poset topology, which connects combinatorics to many other branches of Mathematics [11, 15]. Being more precise, such theory allows us to establish relations between simplicial complexes and finite topological spaces. On the one hand, finite T_0-spaces and finite partially ordered sets are equivalent categories (notice that any finite space is homotopically equivalent to a T_0-space). On the other hand, given a finite topological space X there exists the associated simplicial complex $\mathcal{K}(X)$, where the simplices are its non-empty chains; and, conversely, given a finite simplicial complex K there is a finite space $\chi(K)$, the poset of simplices of K, such that $\mathcal{K}(\chi(K)) = \operatorname{sd} K$, the first barycentric subdivision of K. By these constructions we can see posets and simplicial complexes as essentially equivalent objects.

In this work we introduce a natural notion of LS-category $\operatorname{scat} K$ for any simplicial complex K. Unlike other topological notions established for the geometric realization of the complex, our approach is directly based on the simplicial structure. In this context, *contiguity classes* are the combinatorial analogues of homotopy classes. For instance, different simplicial approximations to the same continuous map are contiguous and the geometric realizations of contiguous maps are homotopic.

Analogously to the topological setting, it is desirable that this notion of category be a homotopy invariant. In order to obtain this goal, the notion of strong collapse introduced by Minian and Barmak [2] is used instead of the classical notion of collapse. The existence of *cores* or *minimal* complexes is a fundamental difference between strong homotopy types and simple homotopy types. A simplicial complex can collapse to non-isomorphic subcomplexes. However if a complex K strongly collapses to a minimal complex K_0, it must be unique, up to isomorphism. We prove the homotopical invariance of simplicial category, and, in particular, that $\operatorname{scat} K = \operatorname{scat} K_0$.

In addition, a notion of *geometric category* $\operatorname{gscat} K$ is introduced in the simplicial context. For topological spaces geometric category is not a homotopical invariant, so it is customary to consider the minimum value of $\operatorname{gcat} Y$, for all spaces Y of the same homotopy type as X. This process leads to a homotopical invariant, $\operatorname{Cat} X$, first introduced by Ganea [5]. In the simplicial context we prove several results about the behaviour of $\operatorname{gscat} K$ under strong collapses. Other authors [1] have considered a notion of geometric category for simple collapses. The essential difference is that for gcat one can consider not only the minimum value in the homotopy class, but also the maximum, which coincides with the category of the *core* of the complex.

By means of the equivalence between simplicial complexes and finite topological spaces, we get a notion of LS-category of finite spaces which corresponds with the classical notion, because the concept of strong homotopy equivalence in the simplicial context corresponds to the notion of homotopy equivalence in the setting of finite spaces. Under this point of view new results are obtained which do not have analogues in the continuous case.

The paper is organized as follows. We start by introducing in Section 2 the basic notions and results concerning the link between simplicial complexes and finite topological spaces, as well as the definition of classical LS-category. Section 3 is focused on the study of the simplicial LS-category $\operatorname{scat} K$ of a simplicial complex K. We prove that this notion is a homotopy invariant, that is, two strongly equivalent complexes have the same category. The corresponding notion of geometrical category $\operatorname{gcat} K$ for a simplicial complex K is studied in Section 4. We obtain that the geometrical category increases under strong collapses, and that the maximum value is obtained for the core K_0 of the complex. Section 5 contains a study on the LS-category of finite topological spaces. Notice that it is not the LS-category of the geometric realization $|\mathcal{K}(X)|$ of the associated simplicial complex, but it is the category of the topological space X itself. We have not found any specific study of LS-category for finite topological spaces in the literature. For instance, we prove that the number of maximal elements minus one is an upper bound of the category of a finite topological space. By analogy with the LS-category of simplicial complexes we establish other results for finite spaces. For instance, we prove that geometrical category increases when a beat point is erased. In particular, we exhibit a new example showing that geometrical category is not a homotopy invariant. This example was communicated to the authors by J. Barmak and G. Minian. Finally, in Section 6 we prove that both the category and the geometrical category decrease when applying the functors $\mathcal{K}$ and χ.

2. Preliminaries

2.1. **Simplicial complexes.** We recall the notions of contiguity and strong collapse. Let K, L be two simplicial complexes. Two simplicial maps $\varphi, \psi \colon K \to L$ are *contiguous* [13, p. 130] if, for any simplex $\sigma \in K$, the set $\varphi(\sigma) \cup \psi(\sigma)$ is a simplex of L; that is, if $v_0, \dots, v_k$ are the vertices of σ then the vertices $f(v_0), \dots, f(v_k), g(v_0), \dots, g(v_k)$ span a simplex of L. This relation, denoted by $\varphi \sim_c \psi$, is reflexive and symmetric, but in general it is not transitive.

Definition 2.1. Two simplicial maps $\varphi, \psi \colon K \to L$ are in the same *contiguity class*, denoted by $\varphi \sim \psi$, if there is a sequence

$$\varphi = \varphi_0 \sim_c \cdots \sim_c \varphi_n = \psi$$

of contiguous simplicial maps $\varphi_i \colon K \to L$, $0 \leq i \leq n$.

A simplicial map $\varphi \colon K \to L$ is a *strong equivalence* if there exists $\psi \colon L \to K$ such that $\psi \circ \varphi \sim \mathrm{id}_K$ and $\varphi \circ \psi \sim \mathrm{id}_L$. We write $K \sim L$ if there is a strong equivalence between the complexes K and L. In the nice paper [3] Barmak and Minian showed that strong homotopy types can be described by certain types of elementary moves called *strong collapses*. A detailed exposition is in Barmak's book [2]. These moves are a particular case of the well known notion of simplicial collapse [7].

Definition 2.2. A vertex v of a simplicial complex K is *dominated* by another vertex $v' \neq v$ if every maximal simplex that contains v also contains v'.

If v is dominated by v' then the inclusion $i \colon K \setminus v \subset K$ is a strong equivalence. Its homotopical inverse is the retraction $r \colon K \to K \setminus v$ which is the identity on $K \setminus v$ and such that $r(v) = v'$. This retraction is called an *elementary strong collapse* from K to $K \setminus v$, denoted by $K \searrow\searrow K \setminus v$.

A *strong collapse* is a finite sequence of elementary collapses. The inverse of a strong collapse is called a strong expansion and two complexes K and L have the same *strong homotopy type* if there is a sequence of strong collapses and strong expansions that transform K into L.

Example 2.3. Figure 1 is an example of elementary strong collapse.

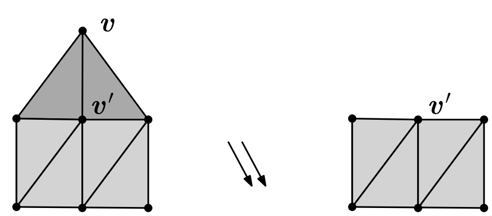

FIGURE 1. Elementary strong collapse

The following result states that the notions of strong homotopy type and strong equivalence (via contiguity) are the same.

Theorem 2.4. [3, Cor. 2.12] *Two complexes K and L have the same strong homotopy type if and only if $K \sim L$.*

2.2. **Finite topological spaces.** We are interested in homotopical properties of finite topological spaces. First, let us recall the correspondence between finite posets and finite T_0-spaces. If $(X, \leq)$ is a partially ordered finite set, we consider the T_0 topology on X given by the basis $\{U_x\}_{x\in X}$ where

$$U_x = \{y \in X : y \leq x\}.$$

Conversely, if (X, τ) is a finite topological space, let U_x be the minimal open set containing $x \in X$. Then we can define a preorder by saying $x \leq y$ if and only if $U_x \subset U_y$. This preorder is an order if and only if τ is T_0. Under this correspondence, a map $f : X \to Y$ between finite T_0-spaces is continuous if and only if f is order preserving. Order spaces are also called "Alexandrov spaces".

Proposition 2.5. *Any (finite) topological space has the homotopy type of a (finite) T_0-space.*

Proof. Take the quotient by the equivalence relation: $x \sim y$ if and only if $U_x = U_y$. □

From now on, we shall deal with finite spaces which are T_0.

Proposition 2.6. [16] *The connected components of X are the equivalence classes of the equivalence relation generated by the order.*

We now consider the notion of homotopy. Let $f, g : X \to Y$ be two continuous maps between finite spaces. We write $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$.

Proposition 2.7. [3] *Two maps $f, g : X \to Y$ between finite spaces are homotopic, denoted by $f \simeq g$, if and only if they are in the same class of the equivalence relation generated by the relation $\leq$ between maps.*

Corollary 2.8. *The basic open sets $U_x \subset X$ are contractible.*

Proof. Since the point x is a maximum of U_x, it is a deformation retract of U_x by means of the constant map $r = x : U_x \to U_x$. □

2.3. **Associated spaces and complexes.** To each finite poset X there is associated the so-called *order complex* $\mathcal{K}(X)$. It is the simplicial complex with vertex set X and whose simplices are given by the finite non-empty chains in the order on X. Moreover, if $f : X \to Y$ is a continuous map, the associated simplicial map $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$ is defined as $\mathcal{K}(f)(x) = f(x)$ for each vertex $x \in X$.

Proposition 2.9. [2, Prop. 2.1.2, Th. 5.2.1]

(1) *If $f, g : X \to Y$ are homotopic maps then the simplicial maps $\mathcal{K}(f)$ and $\mathcal{K}(g)$ are in the same contiguity class.*

(2) *If two T_0-spaces X, Y are homotopy equivalent, then the complexes $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ have the same strong homotopy type.*

Notice that the reciprocal statements are not necessarily true, because two non-homotopic maps f, g may induce maps $\mathcal{K}(f), \mathcal{K}(g)$ which are in the same contiguity class, through simplicial maps which do not preserve order.

Conversely, it is possible to assign to any finite simplicial complex K its *Hasse diagram* or face poset, that is, the poset of simplices of K ordered by inclusion. If $\varphi\colon K \to L$ is a simplicial map, the associated continuous map $\chi(\varphi)\colon \chi(K) \to \chi(L)$ is given by $\chi(\varphi)(\sigma) = \varphi(\sigma)$, for any simplex σ of K.

Proposition 2.10. [2, Prop. 2.1.3, Th.5.2.1]

(1) *If the simplicial maps $\varphi, \psi\colon K \to L$ are in the same contiguity class then the continuous maps $\chi(\varphi), \chi(\psi)$ are homotopic.*
(2) *If two finite simplicial complexes K, L have the same strong homotopy type, then the associated spaces $\chi(K), \chi(L)$ are homotopy equivalent.*

2.4. **LS-category.** We recall the basic definitions of Lusternik-Schnirelmann theory. Well known references are [5] and [10].

An open subset U of a topological space X is called *categorical* if U can be contracted to a point inside the ambient space X. In other words, the inclusion $U \subset X$ is homotopic to some constant map.

Definition 2.11. The *Lusternik-Schnirelmann category*, $\operatorname{cat} X$, of X is the least integer $n \geq 0$ such that there is a cover of X by $n+1$ categorical open subsets. We write $\operatorname{cat} X = \infty$ if such a cover does not exist.

Category is an invariant of homotopy type. Another interesting notion, the *geometric category*, denoted by $\operatorname{gcat} X$, can be defined in a similar way using subsets of X which are contractible in themselves, instead of contractible in the ambient space X. By definition, $\operatorname{cat} X \leq \operatorname{gcat} X$. However, geometric category is not a homotopy invariant [5, p. 79].

Remark 1. For ANRs one can use *closed* covers, instead of open covers, in the definition of LS-category. However, these two notions would lead to different theories in the setting of finite spaces. For instance, for the finite space of Example 5.2, we obtain different values for the corresponding categories. This work is limited to the nowadays most common definition of LS-category, that is, using categorical open subsets.

Remark 2. Actually, the definition of LS-category by covers is not well-suited for many constructions in homotopy theory. This led to alternative definitions (Ganea, Whithead [5]) which are well known in algebraic topology. However, those constructions require that the space X satisfies some additional properties. One of them, the existence of *non-degenerate base-points* is guaranteed by Prop. 2.8. But other properties, like being Hausdorff or even normal, are not satisfied by finite spaces (notice that every finite T_1-space is discrete), so we have not explored them further.

3. LS-category of simplicial complexes

We work in the category of *finite* simplicial complexes and simplicial maps [13]. The key notion introduced in this paper is that of *LS-category* in the simplicial setting. This construction is the natural one when the notion of "homotopy" is that of contiguity class. Contiguous maps were considered in Subsection 2.1.

3.1. **Simplicial category.**

Definition 3.1. Let K be a simplicial complex. We say that the subcomplex $U \subset K$ is *categorical* if there exists a vertex $v \in K$ such that the inclusion $i\colon U \to K$ and the constant map $c_v\colon U \to K$ are in the same contiguity class, $i_U \sim c_v$.

In other words, i factors through v up to "homotopy" (in the sense of contiguity class). Notice that a categorical subcomplex may not be connected.

Definition 3.2. The *simplicial LS-category*, $\operatorname{scat} K$, of the simplicial complex K, is the least integer $m \geq 0$ such that K can be covered by $m+1$ categorical subcomplexes.

For instance, $\operatorname{scat}\ K = 0$ if and only if K has the strong homotopy type of a point.

Example 3.3. The simplicial complex K of Figure 2 appears in [3]. It is collapsible (in the usual sense) but not strongly collapsible, so $\operatorname{scat} K \geq 1$. We can obtain a cover by two strongly collapsible subcomplexes taking a non internal 2-simplex σ and its complement $K \setminus \sigma$. Thus $\operatorname{scat} K = 1$.

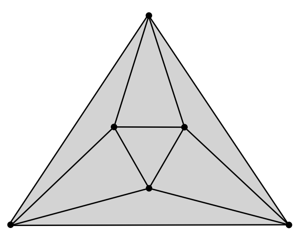

FIGURE 2. A complex K with $\operatorname{scat} K = 1$.

This example shows that scat depends on the simplicial structure more than on the geometric realization of the complex.

3.2. **Homotopical invariance.** The most important property of the simplicial category is that it is an invariant of the strong equivalence type, as we shall prove now.

Theorem 3.4. *Let $K \sim L$ be two strongly equivalent complexes. Then* $\operatorname{scat} K = \operatorname{scat} L$.

We begin with two Lemmas which are easy to prove.

Lemma 3.5. *Let $f, g\colon K \to L$ be two contiguous maps, $f \sim_c g$, and let $i\colon N \to K$ (resp. $r\colon L \to N$) be another simplicial map. Then $f \circ i \sim_c g \circ i$ (resp. $r \circ f \sim_c r \circ g$).*

Lemma 3.6. *Let*

$$K = K_0 \overset{f_1}{\to} K_1 \to \cdots \overset{f_n}{\to} K_n = L$$

and

$$L = K_n \overset{g_n}{\to} \cdots \to K_1 \overset{g_1}{\to} K_0 = K$$

be two sequences of maps such that $g_i \circ f_i \sim_c 1$ and $f_i \circ g_i \sim_c 1$, for all $i \in \{1, \dots, n\}$. Then the complexes K and L are strongly equivalent, $K \sim L$.

The main Theorem 3.4 will be a direct consequence of the following Proposition (by interchanging the roles of K and L).

Proposition 3.7. *Let $f\colon K \to L$ and $g\colon L \to K$ be simplicial maps such that $g \circ f \sim 1_K$. Then $\operatorname{scat} K \leq \operatorname{scat} L$.*

Proof. Let $U \subset L$ be a categorical subcomplex. Since the inclusion i_U is in the contiguity class of some constant map c_v, there exists a sequence of maps $\varphi_i\colon U \to K$, $0 \leq i \leq n$, such that

$$i_U = \varphi_0 \sim_c \cdots \sim_c \varphi_n = c_v.$$

Take the subcomplex $f^{-1}(U) \subset K$. We shall prove that $f^{-1}(U)$ is categorical. Since $g \circ f \sim 1_K$, there is a sequence of maps $\psi_i\colon K \to K$, $0 \leq i \leq m$, such that

$$1_K = \psi_0 \sim_c \cdots \sim_c \psi_m = g \circ f.$$

Denote by f' the restriction of f to $f^{-1}(U)$, with values in U, that is, $f'\colon f^{-1}(U) \to U$, defined by $f'(x) = f(x)$. Denote by $j\colon f^{-1}(U) \subset K$ the inclusion. Then:

$$j = 1_K \circ j = \psi_0 \circ j \sim_c \cdots \sim_c \psi_m \circ j = g \circ f \circ j \tag{1}$$

by Lemma 3.5. Since $f \circ j = i_U \circ f'$, we have

$$g \circ f \circ j = g \circ i_U \circ f' = g \circ \varphi_0 \circ f' \sim_c \cdots \sim_c g \circ \varphi_n \circ f'. \tag{2}$$

But $\varphi_n = c_v$, so $g \circ \varphi_n \circ f'\colon f^{-1}(U) \to g(U)$ is the constant map $c_{g(v)}$. Combining (1) and (2) we obtain

$$j \sim c_{g(v)}.$$

Therefore, the subcomplex $f^{-1}(U) \subset K$ is categorical.

Finally, let $k = \operatorname{scat} L$ and let $\{U_0, \dots, U_k\}$ be a categorical cover of L; then $\{f^{-1}(U_0), \dots, f^{-1}(U_k)\}$ is a categorical cover of K, which shows that $\operatorname{scat} K \leq k$. □

A *core* of a finite simplicial complex K is a subcomplex $K_0 \subset K$ without dominated vertices, such that $K \searrow\searrow K_0$ [3]. Every complex has a core, which is unique up to isomorphism, and two finite simplicial complexes have the same strong homotopy type if and only if their cores are isomorphic.

Since scat is an invariant of the strong homotopy type (Theorem 3.4) we have proved the following result.

Corollary 3.8. *Let K_0 be the core of the simplicial complex K. Then* $\operatorname{scat} K = \operatorname{scat} K_0$.

4. Geometric category

As in the classical case, we shall introduce a notion of *simplicial geometric category* gscat in the simplicial setting, when "homotopy" means to be in the same contiguity class. Another so-called *discrete category*, dcat, which takes into account the notion of *collapsibility* instead of strong collapsibility, has been considered by Scoville et al in [1]. But in contrast with the simplicial LS-category introduced in Section 3, both gscat and dcat are not homotopy invariant. The problem must then be overcome by taking the infimum of the category values over all simplicial complexes which are homotopy equivalent to the given one.

However, our geometric category possesses a remarkable property: due to the notion of *core* complex explained before, there is also a *maximum* of category among the complexes in a given homotopy class.

Remark 3. It is possible to do a translation of the notion of simple collapsibility to finite topological spaces, by means of the notion of *weak* beat point [6].

4.1. **Simplicial geometric category.** According to the notion of strong collapse (defined in Section 2), a simplicial complex K is *strongly collapsible* if it is strongly equivalent to a point. Equivalently, the identity 1_K is in the contiguity class of some constant map $c_v \colon K \to K$.

Definition 4.1. The *simplicial geometric category* $\operatorname{gscat} K$ of the simplicial complex K is the least integer $m \geq 0$ such that K can be covered by $m+1$ strongly collapsible subcomplexes. That is, there exists a cover $U_0, \dots, U_m \subset K$ of K such that $U_i \sim *$, for all $i \in \{0, \dots m\}$.

Notice that strongly collapsible subcomplexes must be connected.

Proposition 4.2. $\operatorname{scat} K \leq \operatorname{gscat} K$.

Proof. The proof is reduced to checking that a strongly collapsible subcomplex is categorical: in fact, the only difference is that in the first case the identity 1_U is in the contiguity class of some constant map c_v, while in the second it is the inclusion $i_U \colon U \to X$ that satisfies $i_U \sim c_v$. □

4.2. **Behaviour under strong collapses.** Obviously, scat and gscat are invariant by simplicial isomorphisms. Moreover we proved in Theorem 3.4 that scat is a homotopy invariant. The next Theorem shows that strong collapses increase the geometric category.

Theorem 4.3. *If L is a strong collapse of K then* $\operatorname{gscat} L \geq \operatorname{gscat} K$.

Proof. Without loss of generality we may assume that there is an elementary strong collapse $r\colon K \to L = K \setminus v$ (see Definition 2.2). If $i\colon L \subset K$ is the inclusion, then $r \circ i = 1_L$ while $\sigma \cup (i \circ r)(\sigma)$ is a simplex of K, for any simplex σ of K. Let V be a strongly collapsible subcomplex of L, that is, the identity 1_V is in the contiguity class of some constant map $c_w\colon V \to V$. That means that there is a sequence of maps $\varphi_i\colon V \to V$, $0 \leq i \leq n$, such that

$$1_V = \varphi_0 \sim_c \cdots \sim_c \varphi_n = c_w.$$

Let us denote $r' = r^{-1}(V) \to V$ the restriction of r to $r^{-1}(V)$, with values in V. Analogously denote $i'\colon V \to r^{-1}(V)$ the inclusion (this is well defined because $r \circ i = 1_V$).

Then, by Lemma 3.5, $\varphi_i \sim_c \varphi_{i+1}$ implies $i' \circ \varphi_i \circ r' \sim_c i' \circ \varphi_{i+1} \circ r'$. Clearly

$$i' \circ \varphi_n \circ r' = i' \circ c_w \circ r' = c_{i(w)}$$

is a constant map. On the other hand it is

$$i' \circ \varphi_0 \circ r' = i' \circ 1_V \circ r' = i' \circ r'$$

and the latter map is contiguous to $1_{r^{-1}(V)}$. This is true because if σ is a simplex of $r^{-1}(V)$ then it is a simplex of K, so $\sigma \cup (i \circ r)(\sigma)$ is a simplex of K, which is contained in $r^{-1}(V)$ because $r \circ i = 1_V$. But $(i \circ r)(\sigma) = (i' \circ r')(\sigma)$, so $\sigma \cup (i' \circ r')(\sigma)$ is a simplex of $r^{-1}(V)$.

We have then proved that the constant map c_w is in the same contiguity class as the identity of $r^{-1}(V)$, which proves that the latter is strongly collapsible.

Now, let $m = \operatorname{gscat} L$ and $\{V_0, \ldots, V_m\}$ a cover of L by strongly collapsible subcomplexes. Then $\{r^{-1}(V_0), \ldots, r^{-1}(V_m)\}$ is a cover of K by strongly collapsible subcomplexes. This proves that $\operatorname{gscat} K \leq m$. □

Remark 4. Example 5.7 about finite spaces and the relations established in Section 6 lead us to thinking that the inequality in the previous Theorem is not an equality. However, we have not found an example of a complex simplicial where the inequality is strict.

Given any finite complex K, by successive elimination of dominated vertices one obtains the core K_0 of the complex K, which is the same for all the complexes in the homotopy class of K. Then we have the following result (compare with Corollary 3.8.

Corollary 4.4. *The geometric category* $\operatorname{gscat} K_0$ *of the core* K_0 *of the complex* K *is the maximum value of* $\operatorname{gscat} L$ *among all the complexes* L *which are strongly equivalent to* K*.*

5. LS-category of finite spaces

In this paper finite posets are considered as topological spaces by themselves, and not as geometrical realizations of its associated order complexes. That is, as emphasized in [2, p. 34], to say that a finite T_0-space X is contractible is different from saying that $|\mathcal{K}(X)|$ is contractible (although X

and $|\mathcal{K}(X)|$ have the same *weak* homotopy type). In this context we shall consider the usual notion of LS-category of topological spaces [5]. We have already introduced it in Definition 2.11.

5.1. **Maximal elements.** The following result establishes an upper bound for the category of a finite poset. Notice that there is not a result of this kind for non-finite topological spaces

Proposition 5.1. *Let $M(X)$ be the number of maximal elements of X. Then* $\operatorname{cat} X \leq \operatorname{gcat} X < M(X)$.

Proof. If $x \in X$ is a maximal element then U_x is contractible (Corollary 2.8), so maximal elements determine a categorical cover. □

In particular, a space with a maximum is contractible, as it is well known.

It is also known that if X has a unique *minimal* element x then X is contractible, because the identity is homotopic to the constant map c_x. Even more, a space X is contractible if and only if its *opposite* space X^{op} (that is, reverse order) is contractible. However, the LS-categories of X and X^{op} may not coincide, as the following Example shows. This is a quick way to check that X and X^{op} are not homotopy equivalent, even if they always are *weak homotopy* equivalent.

Example 5.2. In Figure 3 it is clear that $\operatorname{cat} X = 1$ because X is not contractible and $\operatorname{cat} X < 2$ by Proposition 5.1. However $\operatorname{cat} X^{\mathrm{op}} = 2$ since $\operatorname{cat} X^{\mathrm{op}} < 3$ and it is easy to check that the unions of any two open sets $U_{y_i} \cup U_{y_j}$ are not contractible.

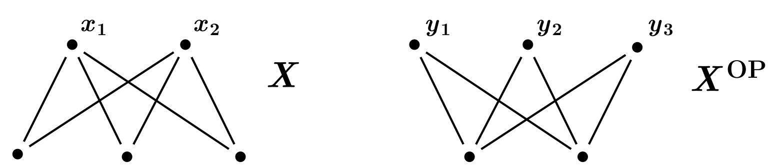

FIGURE 3. A space where $\operatorname{cat} X \neq \operatorname{cat} X^{\mathrm{op}}$.

Notice that for any categorical open cover, the open sets U_x corresponding to maxima must be contained in some element of the cover.

5.2. **Geometric category.** As it was pointed out in Section 2, another homotopy invariant, $\operatorname{Cat} X$, can be defined as the least geometric category of all spaces in the homotopy type of X. A peculiarity of finite topological spaces is that it is also possible to consider the *maximum* value of gcat in each homotopy type. We shall prove that this maximum is attained in the so called *core* space of X, a notion introduced by Stong [14].

The next definition is equivalent to that of linear and collinear points in [14, Th. 2], called *beat points* by other authors [2, 3, 12].

Definition 5.3. Let X be a finite topological space. A point $x_0 \in X$ is a *beat point* if there exists another point $x_0' \neq x_0$ satisfying the following conditions:

(1) If $x_0 < y$ then $x_0' \leq y$;
(2) if $x < x_0$ then $x \leq x_0'$.
(3) x_0 and x_0' are comparable.

In other words, a beat point covers exactly one point or it is covered by exactly one point. Figure 4 shows a beat point with $x_0 \leq x_0'$.

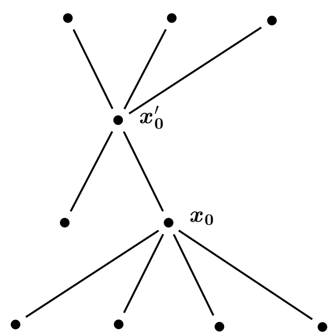

FIGURE 4. An *up* beat point x_0

Proposition 5.4. *If x_0 is a beat point of X then the map $r\colon X \to X \setminus x_0$ given by $r(x) = x$ if $x \neq x_0$ and $r(x_0) = x_0'$, is continuous and satisfies $r \circ i = \mathrm{id}$ and $i \circ r \simeq \mathrm{id}$.*

Corollary 5.5. *If $f\colon X \to X$ is a continuous map such that $f(x_0) = x_0$, then the map g which equals f on $X \setminus x_0$ but sends x_0 onto x_0' is homotopic to f.*

Since $X \setminus x_0$ is a deformation retract of X (Proposition 5.4) it follows that $\operatorname{cat} X \setminus x_0 = \operatorname{cat} X$. However gcat is not a homotopical invariant. The next Theorem shows that geometrical category increases when a beat point is erased.

Theorem 5.6. *If x_0 is a beat point of X then* $\operatorname{gcat} X \setminus x_0 \geq \operatorname{gcat} X$.

Proof. Let $U_0, \dots, U_n$ be a cover of $X \setminus x_0$ such that each U_i is an open subset of $X \setminus x_0$, contractible in itself. We shall define a cover $U_0', \dots, U_n'$ of X as follows.

Let x_0' be a point associated to the beat point x_0 as in Definition 5.3. For each U_i, $0 \leq i \leq n$, we take:

(1) If x_0 is a maximal element of X then $x_0' \leq x_0$ and
   (a) there is some U_i which contains x_0', so we take $U_i' = U_i \cup \{x_0\}$;
   (b) for the other U_j's, if any, we take $U_j' = U_j$.
(2) If x_0 is not a maximal element, then it happens that

(a) for some of the U_i's there exists $y \in U_i$ such that $x_0 < y$; then we take $U'_i = U_i \cup \{x_0\}$;
(b) for the other U_j's, if any, which satisfy $x < x_0$ for all $x \in U_j$, we take $U'_j = U_j$.

Notice that condition (2a) implies that $x'_0 \in U_i$ because $x_0 < y$ implies $x'_0 \leq y$ (by definition of beat point), and U_i is an open subset of $X \setminus x_0$, so the basic open set U_y is contained in U_i.

We shall verify that each U'_i is an open subset of X.

Let $y \in U'_i$ and $x \leq y$. If $x, y \neq x_0$ then $x \in U_i \subset U'_i$ because U_i is an open subset of $X \setminus x_0$. In cases (1a) and (2a), if $y = x_0$ and $x < x_0$ then $x \leq x'_0$ by definition of beat point, and we know that $x'_0 \in U_i$, so we conclude that $x \in U_i \subset U'_i$. Finally, if $x = x_0 < y$ then $x \in U'_i = U_i \cup \{x_0\}$. In cases (1b) and (2b), neither $x_0 \in U'_i$ nor $x = x_0 < y \in U_i$ are possible.

Moreover it is easy to check that x_0 is still a beat point of U'_i, with the same associated point x'_0.

Let $U = U_i$ for some $i \in \{0, \ldots, n\}$; since U is strongly collapsible, the identity map $\mathrm{id}\colon U \to U$ is homotopic to some constant map $c\colon U \to U$. By Proposition 2.7, that means that there is a sequence $\mathrm{id} = \varphi_0, \cdots, \varphi_n = c$ of maps $\varphi_k\colon U \to U$ such that each consecutive pair satisfies either $\varphi_i \leq \varphi_{i+1}$ or $\varphi_i \geq \varphi_{i+1}$.

We shall prove that the identity of $U' = U'_i$ is homotopic to a constant map. Obviously it suffices to consider cases (1a) and (2a). Define $\varphi'_k\colon U' \to U'$ as follows: $\varphi'_k(x) = \varphi_k(x)$ if $x \neq x_0$ and $\varphi'_k(x_0) = \varphi_k(x'_0)$. Thus the maps φ'_k are continuous because φ_k preserves the order, hence φ'_k preserves the order too, as it is easy to check. Moreover if $\varphi_i \leq \varphi_{i+1}$ then $\varphi'_i \leq \varphi'_{i+1}$ (analogously for $\geq$). So we have that $\varphi'_0 \sim \varphi'_n$. Now, the map φ_n is constant, so it is φ'_n. Finally, the map φ'_0 is not the identity, but it is homotopic to the identity by Lemma 5.5.

Finally is is easy to check that the open sets $U'_0, , \ldots, U'_n$ form a cover of X. Since they are contractible, it follows that $\operatorname{gcat} X \leq n$. □

After a finite number of steps, by successive elimination of all the beat points, a *core* or *minimal* space X_0 is obtained, which is in the same homotopy class as X. It is known that this core space is unique up to homeomorphism [14, Th. 4].

Example 5.7. The example in Figure 5, communicated to the authors by J. Barmak and G. Minian, shows that the inequality of Theorem 5.6 can be strict.

On the one hand, since X is not contractible, $\operatorname{gcat} X \geq 1$. In addition, $\{U_a \cup U_b, U_e \cup U_j \cup U_k\}$ is a cover of X by open subsets which are contractible in themselves, so we conclude that $\operatorname{gcat} X = 1$.

On the other hand, let us consider the core X_0 (Figure 6) of the finite space X.

We can observe that $\{U_b, U_j, U_k\}$ is a cover of X_0 by open subsets which are contractible in themselves, so $\operatorname{gcat} X_0 \leq 2$. Finally, we can prove that it

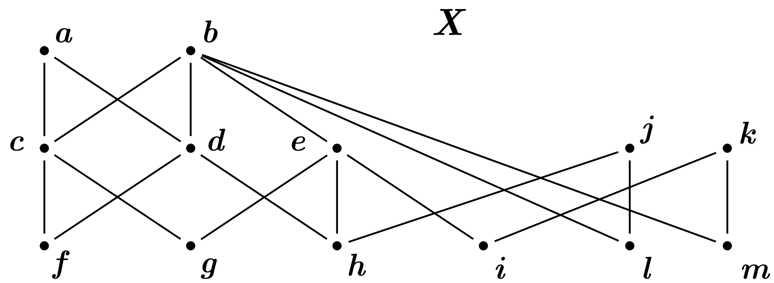

FIGURE 5. A finite space X with $\operatorname{gcat} X_0 > \operatorname{gcat} X$.

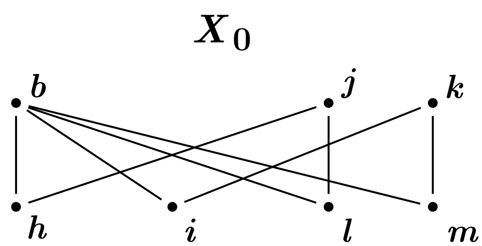

FIGURE 6. The core X_0 of the finite space X in Figure 5.

is not possible to cover X_0 with just two subsets: since each open subset of the cover has to be union of the basic open subsets U_b, U_j, U_k and we note that the unions of two of these open sets are not contractible, we conclude that there is no cover with two elements. Thus $\operatorname{gcat} X_0 = 2$.

Therefore the inequality of Theorem 5.6 is strict for this example.

Remark 5. Notice that X_0 is a new example of a space whose geometrical category does not coincide with its LS-category. It is also a new example showing that the geometrical category of topological spaces is not a homotopy invariant. A classical example is due to Fox [9]. Other examples are given by Clapp and Montejano in [4] , see also Section 3.3 of [5].

The next Corollary is a consequence of Theorem 5.6.

Corollary 5.8. *The geometric category* $\operatorname{gcat} X_0$ *of the core space* X_0 *of* X *equals the maximum of the geometrical categories in its homotopy class.*

Example 5.9. Figure 7 shows a space where $\operatorname{cat} X = 1$ while $\operatorname{gcat} X = 2$. Let us see this. Since X is not contractible, $\operatorname{gcat} X \geq \operatorname{cat} X \geq 1$. Moreover, $\{U_{x_1} \cup U_{x_4}, U_{x_2} \cup U_{x_3}\}$ is a cover of X by categorical open subsets. So we conclude that $\operatorname{cat} X = 1$.

On the other hand, $\{U_{x_1}, U_{x_2} \cup U_{x_3}, U_{x_4}\}$ is a cover of X by open subsets which are contractible in themselves, so $\operatorname{gcat} X \leq 2$. Finally, we can prove that there is no such kind of cover of X with just two subsets: since each open subset of the cover has to be union of basic open subsets U_{x_i}, where x_i are maximal points, and taking into account that the unique union of U_{x_i}'s

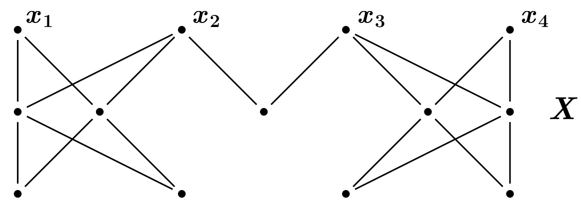

FIGURE 7. A space with $\operatorname{cat} X = 1$ but $\operatorname{gcat} X = 2$.

that is contractible is $U_{x_2} \cup U_{x_3}$, we conclude that it is not possible to get a cover with two elements. Thus $\operatorname{gcat} X = 2$.

6. Relation between categories

We study the relation between the category of a finite T_0-poset X and the simplicial category of the associated order complex $\mathcal{K}(X)$. Analogously, a comparison will be done between the category of a simplicial complex K and its induced Hasse diagram $\chi(K)$. The corresponding definitions were given in Section 2.3.

Proposition 6.1. *Let X be a finite poset and $\mathcal{K}(X)$ its associated order complex. Then* $\operatorname{scat} \mathcal{K}(X) \leq \operatorname{cat} X$*.*

Proof. Let $U_0, \ldots, U_n$ be a categorical cover of X. Then the associated simplicial complexes $\mathcal{K}(U_k)$, $1 \leq k \leq n$, cover $\mathcal{K}(X)$. By definition of LS-category of a topological space (Definition 2.4), each inclusion $i_k \colon U_k \subset X$ is homotopic to some constant map $c_k \colon U_i \to X$, that is, $i_k \simeq c_k$. Then, by Theorem 2.9, the simplicial maps $\mathcal{K}(i_k)$ and $K(c_k)$ from $\mathcal{K}(U_k)$ into $\mathcal{K}(X)$ are in the same contiguity class. Clearly $\mathcal{K}(i_k)$ is the inclusion $\mathcal{K}(U_k) \subset \mathcal{K}(X)$, and $\mathcal{K}(c_k)$ is a constant map. Thus, by definition of LS-category of a simplicial complex (Definition 3.2) the family of complexes $\mathcal{K}(U_k)$ forms a categorical cover of $\mathcal{K}(X)$ and thus $\operatorname{scat} \mathcal{K}(X) \leq n$. $\square$

A completely analogous proof gives the following inequality for the corresponding geometric categories.

Proposition 6.2. $\operatorname{gscat} \mathcal{K}(X) \leq \operatorname{gcat} X$*.*

Example 6.3. Let us consider (Figure 8) the order complex $\mathcal{K}(X)$ of the finite space X of Example 5.9.

Since $\mathcal{K}(X)$ is not strongly collapsible, $\operatorname{gscat} \mathcal{K}(X) \geq 1$. In addition, the two strongly collapsible subcomplexes given in Figure 9 cover $\mathcal{K}(X)$. So we conclude that $\operatorname{gscat} \mathcal{K}(X) = 1$.

It is interesting to point out that the inequality of Proposition 6.2 is strict for this example. However, the upper bound of the Proposition 6.1 is attained since $\operatorname{scat} \mathcal{K}(X) = \operatorname{cat} X = 1$.

Now we shall prove analogous inequalities relating the simplicial category of a finite complex K and the topological category of the face poset $\chi(K)$.

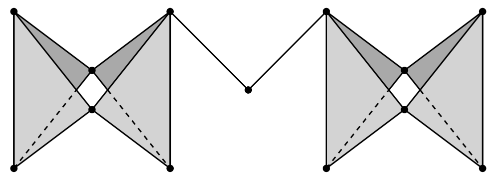

FIGURE 8. The order complex $\mathcal{K}(X)$ of the space X given in Figure 7.

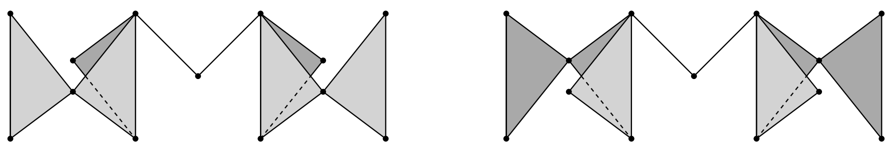

FIGURE 9. Two strongly collapsible subcomplexes

Proposition 6.4. *Let K be a simplicial complex and $\chi(K)$ its Hasse diagram. Then* $\operatorname{cat}\chi(K) \leq \operatorname{scat} K$.

Proof. Let $K_0, \ldots, K_n$ be a cover of K by subcomplexes such that each inclusion $i_k \colon K_k \subset K$ is in the same contiguity class of some constant map $c_k \colon K_k \to K$. Then, using Proposition 2.10, the continuous maps $\chi(i_k)$ and $\chi(c_k)$ are homotopic. By definition (Section 2.3), the first one is the inclusion $\chi(K_k) \subset \chi(K)$, and the second one is a constant map. Then $\chi(K_0), \ldots, \chi(K_n)$ is a categorical cover of $\chi(K)$. Thus $\operatorname{cat}\chi(K) \leq n$. □

A completely analogous proof gives the corresponding result for geometric categories.

Proposition 6.5. $\operatorname{gcat}\chi(K) \leq \operatorname{gscat} K$.

The next Corollary is a direct reformulation in categorical terms of original results due to J. Barmak (Corollary 5.2.8 of [2]).

Corollary 6.6.

(1) $\operatorname{cat} X = 0$ *if and only if* $\operatorname{scat}\mathcal{K}(X) = 0$. *In other words, X is contractible if and only if its order complex $\mathcal{K}(X)$ is strongly collapsible.*
(2) $\operatorname{scat} K = 0$ *if and only if* $\operatorname{cat}\chi(K) = 0$*, that is the complex K is strongly collapsible if and only if its order poset $\chi(K)$ is contractible.*

Finally, we compare the simplicial category of a complex and of its first barycentric subdivision.

Corollary 6.7. *If K is a simplicial complex, then the category of its first barycentric subdivision satisfies* $\operatorname{scat}\operatorname{sd}(K) \leq \operatorname{scat} K$.

Proof. Since $K' = \mathcal{K}(\chi(K))$ equals $\operatorname{sd}(K)$, it follows from Propositions 6.1 and 6.4 that

$$\operatorname{scat} K' \leq \operatorname{cat}\chi(K) \leq \operatorname{scat} K.$$

□

Notice that a complex K and its barycentric subdivision $\mathrm{sd}(K)$ may not have the same strong homotopy type. For instance [2, Example 5.1.13] if K is the boundary of a 2-simplex then both complexes K and $\mathrm{sd}(K)$ do not have beat points. Then if they were in the same homotopy class they would be isomorphic, by Stong's result [14, Th. 3]. But obviously they are not. However, as pointed out in the proof of Corollary 6.6, a complex K is strong collapsible if and only if its barycentric subdivsion $\mathrm{sd}(K)$ is strong collapsible.

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